Geometrical techniques for the N-dimensional Quantum Euclidean Spaces [*]

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Abstract

We briefly report our application [1] of a version of noncommutative geometry to the quantum Euclidean space $\boldsymbol{R}_q^N$, for any $N \geq 3$; this space is covariant under the action of the quantum group $SO_q(N)$, and two covariant differential calculi are known on it. More precisely, we describe how to construct in a Cartan 'moving-frame formalism' the metric, two covariant derivatives, the Dirac operator, the frame, the inner derivations dual to the frame elements, for both of these calculi. The components of the frame elements in the basis of differentials provide a 'local realization' of the Faddeev-Reshetikhin-Takhtadjan generators of $U_q^{\pm}(so(N))$.

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1 Introduction

The idea that the structure of space-time at short distances may be well described by a non-commutative geometry has been appealing since 1947 [2], because such noncommutativity might lead to a regularization of the corresponding field theory (see e.g. [3], [4], [5]). Here we apply the formalism of noncommutative geometry [6], [7] to the quantum Euclidean spaces $\mathbb{R}_q^N$ with $N \geq 3$ [8], which are comodule algebras of the quantum groups $SO_q(N)$. To achieve this goal we use a noncommutative generalization [9] of the moving-frame formalism of E. Cartan. We generalize the results which had been previously found [10] for $\mathbb{R}_q^3$.

When N is odd we can follow a scheme similar to the one developped for $N = 3$. For each of the two $SO_q(N)$ covariant differential calculi defined on $\mathbb{R}_q^N$ we find two torsion free covariant derivatives. After adding to the algebra a 'dilatator' Λ, it is possible to construct an (essentially) unique metric, in such a way that the covariant derivative is compatible with it. By further enlarging the algebra by the square roots and inverses of some elements, we are also able to find for each of the calculi a frame and the derivatives dual to it. When N is even, it is necessary to add also one of the components K of the angular momentum. Some of the elements we add have vanishing derivative but are none-the-less noncommutative analogues of non-constant functions. Then their inclusion can be interpreted as an embedding of the 'configuration space' into part of 'phase space'.

In Section 2 we briefly recall the tools of noncommutative geometry [6] which will be needed. We start with a formal noncommutative algebra $\mathcal{A}$ and with a differential calculus $\Omega^*(\mathcal{A})$ over it, and define then the concepts of a frame or 'Stehbein' [9], the corresponding metric, covariant derivative, and generalized Dirac operator [6]. In Section 3 we shortly review the definiton of $\mathbb{R}_q^N$ [8] and then the construction of two $SO_q(N)$-covariant differential calculi [11, 12, 13] on $\mathbb{R}_q^N$ based on the $\hat{R}$-matrix formalism. Both yield the de Rham calculus in the commutative limit. In Section 4 we proceed with the actual construction of the frame over $\mathbb{R}_q^N$ and of the inner derivations dual to it. Within this framework we recover the 'Dirac operator', which had already been found [14, 15]. We then determine of the metric and the covariant derivatives.

It turns out that the components of the frame in the ξ^i basis automatically provide a 'local realization' of $U_q^\pm(so(N))$ in the extended algebra of $\mathbb{R}_q^N$, i.e. they satisfy the 'RLL' and the 'gLL' relations fulfilled by the $\mathcal{L}^\pm$ [8] generators of $U_q^\pm(so(N))$ and also fulfil the commutation relations of the latter generators with the coordinates x^i. In the case of odd N it is possible to 'glue' them together to get a realization of the whole of $U_q^\pm(so(N))$.

2 The Cartan moving-frame formalism

We start by reviewing a noncommutative extension [9] of the moving-frame formalism of E. Cartan. The building blocks are a noncommutative algebra $\mathcal{A}$,

which in the commutative limit should become the algebra of functions on a parallelizable manifold M, and a differential calculus $\Omega^*(\mathcal{A})$ on it, which should reduce to the ordinary de Rham differential calculus on M in the same limit. The module of the 1-forms $\Omega^1(\mathcal{A})$ is required to be free of rank N (N dimension of the manifold), so that it admits a special basis $\{\theta^a\}_{1 \leq a \leq N}$, referred to as 'frame' or 'Stehbein', which commutes with the elements of $\mathcal{A}$:

$$[f, \theta^a] = 0. \tag{1}$$

We suppose that the basis θ^a is dual to a set of inner derivations $e_a = \operatorname{ad} \lambda_a$:

$$df = e_a f \theta^a = [\lambda_a, f] \theta^a \tag{2}$$

for any $f \in \mathcal{A}$. Then it is possible to find a formal 'Dirac operator' [6]

$$\theta = -\lambda_a \theta^a, \text{ such that } df = -[\theta, f]. \tag{3}$$

We shall require the center $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$ to be trivial: $\mathcal{Z}(\mathcal{A}) = \mathbb{C}$, if this condition is not verified, we shall enlarge the algebra until it does. If we define the (wedge) product π in $\Omega^*(\mathcal{A})$ by relations of the form

$$\theta^a \theta^b = P^{ab}{}_{cd} \theta^c \otimes \theta^d, \qquad P^{ab}{}_{cd} \in \mathcal{Z}(\mathcal{A}) \tag{4}$$

then the λ_a have to satisfy a quadratic relation of the form

$$2\lambda_c \lambda_d P^{cd}{}_{ab} - \lambda_c F^c{}_{ab} - K_{ab} = 0., \qquad F^c{}_{ab}, \ K_{ab} \in \mathcal{Z}(\mathcal{A}) \tag{5}$$

In the case of the quantum Euclidean spaces $\mathbb{R}^N_q$ it turns out that $F^c{}_{ab}, K_{ab} = 0$.

Next, the metric is defined as a nondegenerate $\mathcal{A}$-bilinear map

$$g : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \mathcal{A}, \tag{6}$$

We shall denote

$$g(\theta^a \otimes \theta^b) = g^{ab}. \tag{7}$$

As a further step, a 'generalized flip' can be introduced, an $\mathcal{A}$-bilinear map

$$\sigma : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}), \qquad \sigma(\theta^a \otimes \theta^b) = S^{ab}{}_{cd} \theta^c \otimes \theta^d. \tag{8}$$

Due to bilinearity $g^{ab} \in \mathcal{Z}(\mathcal{A}) = \mathbb{C}$ and $S^{ab}{}_{cd} \in \mathcal{Z}(\mathcal{A}) = \mathbb{C}$.

The flip is necessary in order to construct a covariant derivative D [16], i.e. a map

$$D : \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes \Omega^1(\mathcal{A}) \tag{9}$$

satisfying a left and right Leibniz rule:

$$D(f\xi) = df \otimes \xi + f D\xi, \qquad D(\xi f) = \sigma(\xi \otimes df) + (D\xi)f. \tag{10}$$

Then the torsion map can be consistently be defined as

$$\Theta : \Omega^1(\mathcal{A}) \to \Omega^2(\mathcal{A}), \qquad \Theta = d - \pi \circ D. \tag{11}$$

where bilinearity requires that

$$\pi \circ (\sigma + 1) = 0. \tag{12}$$

The curvature map associated to D is defined by

$$\mathrm{Curv} \equiv D^2 = \pi_{12} \circ D_2 \circ D, \qquad \mathrm{Curv}(\theta^a) = -\frac{1}{2} R^a{}_{bcd} \theta^c \theta^d \otimes \theta^b. \tag{13}$$

Here D_2 is a natural continuation of the map (8) to the tensor product $\Omega^1(\mathcal{A}) \otimes_\mathcal{A} \Omega^1(\mathcal{A})$, namely $D_2(\xi \otimes \eta) = D\xi \otimes \eta + \sigma_{12}(\xi \otimes D\eta)$.

If $F^a{}_{bc} = 0$ a torsion-free covariant derivative [16] is

$$D\xi = -\theta \otimes \xi + \sigma(\xi \otimes \theta). \tag{14}$$

The compatibility of a covariant derivative with the metric is [17]

$$g_{23} \circ D_2 = d \circ g, \qquad S^{ae}{}_{df} g^{fg} S^{bc}{}_{eg} = g^{ab} \delta^c_d. \tag{15}$$

We suppose [18] that σ satisfies the braid relation.

3   The quantum Euclidean spaces

In this section some basic results about the N-dimensional quantum Euclidean space $\mathbb{R}^N_q$ due to [8] are reviewed. We start with the matrix $\hat{R}$ [8] for $SO_q(N, \mathbb{C})$. It is a symmetric $N^2 \times N^2$ matrix, and its main property is that it satisfies the braid relation. It admits [8] a projector decomposition:

$$\hat{R} = q\mathcal{P}_s - q^{-1}\mathcal{P}_a + q^{1-N}\mathcal{P}_t. \tag{16}$$

where the $\mathcal{P}_s$, $\mathcal{P}_a$, $\mathcal{P}_t$ are $SO_q(N)$-covariant q-deformations of the symmetric trace-free, antisymmetric and trace projectors respectively. The projector $\mathcal{P}_t$ projects on a one-dimensional sub-space and can be written in the form $\mathcal{P}_t{}^{ij}_{kl} = (g^{sm}g_{sm})^{-1}g^{ij}g_{kl}$. This leads to the definition of a metric matrix. It is a $N \times N$ matrix g_{ij}, which is a $SO_q(N)$-isotropic tensor and is a deformation of the ordinary Euclidean metric $g_{ij} = q^{-\rho_i}\delta_{i,-j}$. If n is the rank of $SO(N, \mathbb{C})$, the indices take the values $i = -n, \ldots, -1, 0, 1, \ldots n$ for N odd, and $i = -n, \ldots, -1, 1, \ldots n$ for N even. Moreover, we have introduced the notation $\rho_i = (n - \frac{1}{2}, \ldots, \frac{1}{2}, 0, -\frac{1}{2}, \ldots, \frac{1}{2} - n)$ for N odd, $(n-1, \ldots, 0, 0, \ldots, 1 - n)$ for N even.

The metric and the braid matrix satisfy the 'gTT' relations [8]

$$g_{il} \hat{R}^{\pm 1 lh}{}_{jk} = \hat{R}^{\mp 1 hl}{}_{ij} g_{lk}, \qquad g^{il} \hat{R}^{\pm 1 jk}{}_{lh} = \hat{R}^{\mp 1 ij}{}_{hl} g^{lk}. \tag{17}$$

With the help of the projector $\mathcal{P}_a$, the N-dimensional quantum Euclidean space is defined as the associative algebra $\mathbb{R}^N_q$ generated by elements $\{x^i\}_{i=-n,\ldots,n}$ with relations

$$\mathcal{P}_a{}^{ij}_{kl} x^k x^l = 0. \tag{18}$$

or, more explicitly [12]

$$x^i x^j = q x^j x^i \text{ for } i < j, i \neq -j, \qquad [x^i, x^{-i}] = \begin{cases} k\omega_{i-1}^{-1} r_{i-1}^2 & \text{for } i > 1 \\ 0 & \text{for } i = 1, N \text{ even}, \\ hr_0^2 & \text{for } i = 1, N \text{ odd}. \end{cases}$$
(19)

We use the notation here $\omega_i = q^{\rho_i} + q^{-\rho_i}$, $h = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$, $k = q^{\frac{1}{2}} - q^{-\frac{1}{2}}$ and

$$r_i^2 = \sum_{k,l=-i}^{i} g_{kl} x^k x^l, \qquad i \geq 0 \text{ for } N \text{ odd}, i \geq 1 \text{ for } N \text{ even}.$$
(20)

For $q \in \mathbb{R}^+$ a conjugation $(x^i)^* = x^j g_{ji}$ can be defined on $\mathbb{R}_q^N$ to obtain what is known as quantum real Euclidean space $\mathbb{R}_q^N$.

As this will be necessary for the construction of the elements λ_a, we enlarge the algebra $\mathbb{R}_q^N$ with the real elements $r_i^{\pm 1} = (r_i^2)^{\pm \frac{1}{2}}$, $i = 0 \ldots n$. There is a unique way to postulate their commutation relations with x^j so that the latter give the commutation relations between r_i^2 and x^j which can be drawn from (18). Namely

$$x^j r_i = r_i x^j \text{ for } |j| \leq i, \qquad x^j r_i = q r_i x^j \text{ for } j < -i, \qquad x^j r_i = q^{-1} r_i x^j \text{ for } j > i$$
(21)

Note that $r \equiv r_n$ turns out to be central.

There are [11] two quadratic differential calculi $\Omega^*(\mathbb{R}_q^N)$ and $\bar{\Omega}^*(\mathbb{R}_q^N)$, which are covariant with respect to $SO_q(N)$.

$$x^i \xi^j = q \, \hat{R}_{kl}^{ij} \xi^k x^l, \qquad \mathcal{P}_{s,t}{}_{kl}^{ij} \xi^k \xi^l = 0 \quad \text{for } \Omega^1(\mathbb{R}_q^N), \tag{22}$$

$$x^i \bar{\xi}^j = q^{-1} \, \hat{R}^{-1}{}_{kl}^{ij} \bar{\xi}^k x^l, \qquad \mathcal{P}_{s,t}{}_{kl}^{ij} \bar{\xi}^k \bar{\xi}^l = 0 \quad \text{for } \bar{\Omega}^1(\mathbb{R}_q^N), \tag{23}$$

where $dx^i = \xi^i$ and $\bar{d}x^i = \bar{\xi}^i$. If a $*$-structure on $\Omega^1(\mathbb{R}_q^N) \oplus \bar{\Omega}^1(\mathbb{R}_q^N)$ is defined by setting $(\xi^i)^* = \bar{\xi}^j g_{ji}$, the two calculi are seen to be conjugate.

The Dirac operator [6] of 3 is easily verified to be given by [15, 14].

$$\theta \quad = \omega_n q^{\frac{N}{2}} k^{-1} r^{-2} g_{ij} x^i \xi^j, \quad \text{for } \Omega^1(\mathbb{R}_q^N), \tag{24}$$

$$\bar{\theta} \quad = -\omega_n q^{-\frac{N}{2}} k^{-1} r^{-2} g_{ij} x^i \bar{\xi}^j \quad \text{for } \Omega^1(\mathbb{R}_q^N). \tag{25}$$

Now, we have the following difficulty. In Section 2 we required the center of the algebra $\mathcal{A}$ to be trivial. But the algebra generated by the x^i and r_j has a nontrivial center, therefore the formalism cannot be directly applied to it. With a general Ansatz of the type $\theta^a = \theta_i^a \xi^i$, we immediately see that the condition (1) cannot be fulfilled for r^2 if $r_n^2 \in \mathcal{Z}(\mathcal{A})$. To solve this problem we add to the algebra also a unitary element Λ, and its inverse Λ^{-1}. It is a "dilatator", which satisfies the commutation relations

$$x^i \Lambda = q \Lambda x^i. \tag{26}$$

But this is not enough in the case of even N. We have added the elements $r_1^{\pm 1} = (x^{-1} x^1)^{\pm \frac{1}{2}}$ and therefore the center is non trivial even after Λ has been

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added, because the elements $r_1^{-1}x^{\pm 1}$ commute also with Λ. We choose to add a Drinfeld-Jimbo generator $K = q^{\frac{H_1}{2}}$ and its inverse K^{-1}, where H_1 belongs to the Cartan subalgebra of $U_q so(N)$ and represents the component of the angular momentum in the $(-1, 1)$-plane. This new element satisfies the commutation relations

$$K\Lambda = \Lambda k, \qquad Kx^{\pm 1} = q^{\pm 1}x^{\pm 1}K, \qquad Kx^{\pm i} = x^{\pm i}K \text{ for } i > 1 \qquad (27)$$

There are many ways to fix the commutation relations of Λ with the 1-forms compatibly with (26). We choose [12]

$$\xi^i \Lambda = \Lambda \xi^i, \qquad \Lambda d = q d\Lambda. \qquad (28)$$

This choice has the disadvantage that Λ does not satisfy the Leibniz rule $d(fg) = f dg + (df)g \; \forall f, g \in \mathbb{R}_q^N$. Nevertheless, Λ can then be interpreted in a consistent way as an element of the Heisenberg algebra, because Λ^{-2} can be constructed [12] as a simple polynomial in the coordinates and derivatives, and, moreover, in the next section we shall see that this allows us to normalize the θ^a and λ_a in such a way as to recover $\mathbb{R}^N$ as geometry in the commutative limit.

As already observed [10] there are other possibilities, e.g. we could have set $d\Lambda = 0$. This choice, however, is not completely satisfactory neither, because we would like $df = 0$ to hold only for the analogues of the constant functions and, moreover, with a procedure similar to the one described previously [10] for $N = 3$, we would recover as geometry in the commutative limit $\mathbb{R} \times S^{N-1}$ rather than $\mathbb{R}^N$.

The same discussion which hold for Λ can be done to determine the commutation relations between K and the 1-forms ξ^i. We choose $dK = 0$. Then consistency with (27) requires

$$K\xi^1 = q^{\pm 1}\xi^{\pm 1}K, \qquad K\xi^i = \xi^i K \quad \text{ for } i > 1. \qquad (29)$$

4 Inner derivations, frame, metric and covariant derivatives

Now, we would like to proceed with the actual construction of a frame θ^a and of the the associated inner derivations $e_a = \text{ad}\,\lambda_a$ satisfying the conditions in Section 2 for the extended algebra of $\mathbb{R}_q^N$. We start with the Ansatz

$$\theta^a = \theta_i^a \xi^i \qquad (30)$$

for θ^a, but allow the coefficients θ_i^a to depend on Λ. The equation $[r, \theta^a] = 0$ fixes the dependence of the frame on the dilatator to be linear in Λ^{-1}. From the duality condition (2) one sees immediately that the matrices

$$e_a^i = [\lambda_a, x^i] \qquad (31)$$

must be inverse to θ_i^a in the sense that $e_a^i \theta_j^a = \delta_j^i$, $\theta_i^a e_b^i = \delta_b^a$.

Equation (1) is equivalent to

$$x^i \theta_j^a = q^{-1} \hat{R}^{-1}{}_{lj}^{ki} \theta_k^a x^l, \qquad x^h e_a^i = q \hat{R}_{jk}^{hi} e_a^j x^k. \tag{32}$$

As the λ_a have each only one index, while the coefficients θ_i^a of the frame have two, it is easier to look for solutions λ_a to the equation

$$x^h [\lambda_a, x^i] = q \hat{R}_{jk}^{hi} [\lambda_a, x^j] x^k. \tag{33}$$

The inner derivations e_a^i can be easily computed as commutators of the λ_a with the coordinates according to (31), the matrix θ_i^a can be recovered as the inverse of e_a^i.

The r-dependence of θ^a is fixed by their commutation relations with Λ. We shall require

$$[\Lambda, \theta^a] = 0 \implies [\theta_i^a, \Lambda] = 0, \qquad [e_a^i, \Lambda] = 0. \tag{34}$$

Our main results are the following theorems [1].

Theorem 1 *N independent solutions of Equation (33) are given by*

$$\begin{aligned} &\lambda_0 = \gamma_0 \Lambda (x^0)^{-1} \text{ for } N \text{ odd,} \quad \lambda_{\pm 1} = \gamma_{\pm 1} \Lambda (x^{\pm 1})^{-1} K^{\mp 1} \text{ for } N \text{ even,} \\ &\lambda_a = \gamma_a \Lambda r_{|a|}^{-1} r_{|a|-1}^{-1} x^{-a} \qquad \qquad \text{otherwise,} \end{aligned} \tag{35}$$

where $\gamma_a \in \mathbb{C}$ are arbitrary normalization constants.

This has been proven by a direct computation in [1]. The proof is too long to write it here.

Theorem 2 *If the normalization constants in theorem 1 satisfy the conditions*

$$\begin{aligned} \gamma_0 &= -q^{-\frac{1}{2}} h^{-1} & & \text{for } N \text{ odd,} \\ \gamma_1 \gamma_{-1} &= \begin{cases} -q^{-1} h^{-2} & \text{for } N \text{ odd,} \\ k^{-2} & \text{for } N \text{ even,} \end{cases} \\ \gamma_a \gamma_{-a} &= -q^{-1} k^{-2} \omega_a \omega_{a-1} & & \text{for } a > 1. \end{aligned} \tag{36}$$

then the elements λ_a fulfill among themselves the commutation relations

$$\mathcal{P}_{acd}^{ab} \lambda_a \lambda_b = 0 \tag{37}$$

and the matrices $e_a^i = [\lambda_a, x^i]$ satisfy

$$\begin{aligned} RTT-relations: \qquad & \hat{R}_{kl}^{ij} e_a^k e_b^l = e_c^i e_d^j \hat{R}_{ab}^{cd} & (38) \\ gTT-relations: \qquad & g^{ab} e_a^i e_b^j = g^{ij} \Lambda^2, \qquad g_{ij} e_a^i e_b^j = g_{ab} \Lambda^2 & (39) \\ normalization: \qquad & e_0^0 e_0^0 = \Lambda^2. & (40) \end{aligned}$$

Again, the proof would be too long and it can been found in [1].

Let us make some remarks. It is interesting to note that the commutation relations (37) between the λ_a are the same as those (19) satisfied by the x^i, and therefore the linear and constant terms in (5) vanish.

The relations (36) fix only the value of the product $\gamma_a \gamma_{-a}$. The determination of it can be done e.g. by applying the gTT-relations for $i = -j$. We see that γ_0^2 for N odd and $\gamma_1 \gamma_{-1}$ for N even are positive real numbers, while all the remaining products $\gamma_a \gamma_{-a}$ are negative.

Now, an analogous construction can be done for the barred calculus $\bar{\Omega}^*(\mathcal{A})$.

Theorem 3 *N independent solutions of Equation*

$$[\bar{\lambda}_a, x^i]x^j = q^{-1}\hat{R}^{-1}{}^{li}_{kj}x^l[\bar{\lambda}_a, x^k]. \tag{41}$$

are given by

$$\bar{\lambda}_0 = \bar{\gamma}_0 \Lambda^{-1}(x^0)^{-1} \text{ for } N \text{ odd}, \quad \bar{\lambda}_{\pm 1} = \bar{\gamma}_{\pm 1}\Lambda^{-1}(x^{\pm 1})^{-1}K^{\pm 1} \text{ for } N \text{ even},$$

$$\bar{\lambda}_a = \bar{\gamma}_a \Lambda^{-1}r^{-1}_{|a|}r^{-1}_{|a|-1}x^{-a} \qquad \text{otherwise}, \tag{42}$$

where $\bar{\gamma}_a \in \mathbb{C}$ are arbitrary normalization constants.

Theorem 4 *If the normalization constants in theorem 3 satisfy the conditions*

$$\bar{\gamma}_0 = q^{\frac{1}{2}}h^{-1} \qquad\qquad \text{for } N \text{ odd},$$

$$\bar{\gamma}_1 \bar{\gamma}_{-1} = \begin{cases} -qh^{-2} & \text{for } N \text{ odd}, \\ k^{-2} & \text{for } N \text{ even}, \end{cases} \tag{43}$$

$$\bar{\gamma}_a \bar{\gamma}_{-a} = -qk^{-2}\omega_a \omega_{a-1} \quad \text{for } a > 1,$$

then the elements $\bar{\lambda}_a$ fulfill among themselves the commutation relations

$$\mathcal{P}^{ab}_{acd}\bar{\lambda}_a \bar{\lambda}_b = 0 \tag{44}$$

and the matrices $\bar{e}^i_a = [\bar{\lambda}_a, x^i]$ satisfy the

$$RTT - relations: \qquad \hat{R}^{ij}_{kl}\bar{e}^k_a \bar{e}^l_b = \bar{e}^i_c \bar{e}^j_d \hat{R}^{cd}_{ab} \tag{45}$$

$$gTT - relations: \qquad g^{ab}\bar{e}^i_a \bar{e}^j_b = g^{ij}\Lambda^{-2}, \qquad g_{ij}\bar{e}^i_a \bar{e}^j_b = g_{ab}\Lambda^{-2}, \tag{46}$$

$$normalization: \qquad \bar{e}^0_0 \bar{e}^0_0 = \Lambda^{-2}. \tag{47}$$

The conditions (38),(39), (40) in theorem 2 and (45), (46), (47) in theorem 4 are equivalent to the defining relations satisfied by the generators $\mathcal{L}^{\pm i}_a$ [8] of $U^{\pm}_q(so(N))$, i.e. we have found a 'local realization' of the two Borel subalgebras $U^{\pm}_q(so(N))$ of $U_q(so(N))$. We can ask, under which circumstances we can 'glue' them together to construct a realization of the whole of $U_q(so(N))$. The answer is given by the following theorem [1].

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Theorem 5 *In the case of odd N with the $\lambda_j, \bar{\lambda}_j$ defined as in (35) and (42) and with coefficients given by*

$$
\begin{array}{ll}
\gamma_0 = -q^{-\frac{1}{2}} h^{-1}, & \gamma_1^2 = -q^{-2} h^{-2}, \\
\gamma_a^2 = -q^{-2} \omega_a \omega_{a-1} k^{-2} \quad \text{for } a > 1, \quad \gamma_a = q \gamma_{-a} \quad \text{for } a \leq 1, & \quad (48) \\
\bar{\gamma}_a = -q \gamma_a &
\end{array}
$$

then the matrices e_a^i, $\bar{e}_a^i$ satisfy the relations

$$
e_i^i \bar{e}_i^i = 1 \quad \text{(no sum over i)}, \qquad \hat{R}_{ab}^{cd} \bar{e}_c^i e_d^j = \hat{R}_{kl}^{ij} e_a^k \bar{e}_b^l \qquad (49)
$$

The $\gamma_a, \bar{\gamma}_a$ for $a \neq 0$ are imaginary and fixed only up to a sign. This has as a consequence that the homomorphism φ does not preserve the star structure of $U_q(so(N))$. It can be proven [1] that it is not possible to extend this theorem to the case of even N, because it is not possible to verify (49).

For the frames $\theta^a, \bar{\theta}^a$ we find:

$$
\theta^a = \theta_l^a \xi^l = \Lambda^{-2} g^{ab} [\lambda_b, x^j] g_{jl} \xi^l, \qquad \bar{\theta}^a = \bar{\theta}_l^a \bar{\xi}^l = \Lambda^2 g^{ab} [\bar{\lambda}_b, x^j] g_{jl} \bar{\xi}^l. \qquad (50)
$$

They commute both with the coordinates and with Λ and the matrix elements $\bar{\theta}_i^a, \theta_i^a$ fulfill

$$
\hat{R}_{cd}^{ab} \theta_j^d \theta_i^c = \theta_l^b \theta_k^a \hat{R}_{ij}^{kl}, \qquad \hat{R}_{cd}^{ab} \bar{\theta}_j^d \bar{\theta}_i^c = \bar{\theta}_l^b \bar{\theta}_k^a \hat{R}_{ij}^{kl}, \qquad (51)
$$

$$
g_{ab} \theta_j^b \theta_i^a = \Lambda^{-2} g^{ij}, \quad g^{ij} \theta_j^b \theta_i^a = \Lambda^{-2} g_{ab}, \quad g_{ab} \bar{\theta}_j^b \bar{\theta}_i^a = \Lambda^2 g^{ij}, \quad g^{ij} \bar{\theta}_j^b \bar{\theta}_i^a = \Lambda^2 g_{ab}. (52)
$$

This implies that

$$
\mathcal{P}_{s,t}{}_{cd}^{ab} \theta^c \theta^d = 0, \qquad \mathcal{P}_{s,t}{}_{cd}^{ab} \bar{\theta}^c \bar{\theta}^d = 0. \qquad (53)
$$

In other words, the θ^a, $\bar{\theta}^a$ satisfy the same commutation relations as the $\xi^a, \bar{\xi}^a$.

Finally we summarize the results found in [1] for the metric and the covariant derivative for each of the two calculi $\Omega(\mathbb{R}_q^N)$ and $\Omega^*(\mathbb{R}_q^N)$. In the θ^a, $\bar{\theta}^a$ basis respectively the actions of g and σ are

$$
\sigma(\theta^a \otimes \theta^b) = S^{ab}{}_{cd} \theta^c \otimes \theta^d, \quad g(\theta^a \otimes \theta^b) = g^{ab} \quad \text{for } \Omega^*(\mathbb{R}_q^N), \qquad (54)
$$

$$
\sigma(\bar{\theta}^a \otimes \bar{\theta}^b) = \bar{S}^{ab}{}_{cd} \bar{\theta}^c \otimes \bar{\theta}^d, \quad g(\bar{\theta}^a \otimes \bar{\theta}^b) = g^{ab} \quad \text{for } \bar{\Omega}^*(\mathbb{R}_q^N). \qquad (55)
$$

Unfortunately, it is not possible to satisfy simultaneously the metric compatibility condition (15) and the bilinearity condition for the torsion (12). The best we can do is to weaken the compatibility condition to a condition of proportionality. Then for each calculus we find the two solutions for σ:

$$
S = q\hat{R}, \quad S = (q\hat{R})^{-1} \quad \text{for } \Omega^*(\mathbb{R}_q^N), \qquad (56)
$$

$$
\bar{S} = q\hat{R}, \quad \bar{S} = (q\hat{R})^{-1} \quad \text{for } \bar{\Omega}^*(\mathbb{R}_q^N). \qquad (57)
$$

This implies that the covariant derivatives and metric are compatible only up to a conformal factor:

$$
S_{df}^{ae} g^{fg} S_{eg}^{cb} = q^{\pm 2} g^{ac} \delta_d^b, \qquad \bar{S}^{ae}{}_{df} g^{fg} \bar{S}^{cb}{}_{eg} = q^{\pm 2} g^{ac} \delta_d^b. \qquad (58)
$$

In the ξ^i, $\bar{\xi}^i$ basis the actions of g and σ become

$$g(\xi^i \otimes \xi^j) = g^{ij}\Lambda^2 \quad \sigma(\xi^i \otimes \xi^j) = S_{hk}^{ij}\xi^h \otimes \xi^k \quad \text{for } \Omega^*(\mathbb{R}_q^N), \quad (59)$$

$$g(\bar{\xi}^i \otimes \bar{\xi}^j) = g^{ij}\Lambda^{-2}, \quad \sigma(\bar{\xi}^i \otimes \bar{\xi}^j) = \bar{S}^{ij}{}_{hk}\bar{\xi}^h \otimes \bar{\xi}^k. \quad \text{for } \bar{\Omega}^*(\mathbb{R}_q^N). \quad (60)$$

According to (14) the two associated covariant derivatives, one for each choice of σ, are

$$D\xi = -\theta \otimes \xi + \sigma(\xi \otimes \theta), \qquad \bar{D}\bar{\xi} = -\bar{\theta} \otimes \bar{\xi} + \sigma(\bar{\xi} \otimes \bar{\theta}). \quad (61)$$

The associated linear curvatures Curv and $\overline{\text{Curv}}$ vanish, as was to be expected, because $\mathbb{R}_q^N$ should be flat.

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